Promote proportional reasoning through digital technology

Promover el razonamiento proporcional mediante la tecnología digital

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Keywords
Proportional reasoning; mathematics; digital technology; high school education; didactics

ABSTRACT

This article explores alternative teaching methods that promote proportional reasoning in Mexican students aged fourteen to fifteen using digital technology. To this end, a sequence of didactic activities has been designed to give meaning to the concepts of ratio and proportion in their different representations from the approach of linear functions. For the design of the tasks, elements of the theory of Realistic Mathematics Education and the Didactics of CuevasPluvinage were used, coinciding with several didactic points. The proposed learning objectives and activities have been organized through a hypothetical learning trajectory. The analysis and detailed evaluation of the responses allowed us to identify: the didactic advantages of the design, the learning difficulties, and the vital role played by technology in the learning process.

Palabras clave
Razonamiento proporcional; matemáticas; tecnología digital; educación secundaria; didáctica

RESUMEN

En este artículo se exploran formas alternativas de enseñanza que promuevan el razonamiento proporcional en estudiantes mexicanos entre catorce y quince años, con apoyo de la tecnología digital. Con este propósito se diseñó una secuencia de actividades didácticas para significar los conceptos de razón y proporción en sus diversas representaciones desde la perspectiva de las funciones lineales. Para el diseño de tareas se emplearon elementos del marco de la matemática realista y la didáctica de Cuevas-Pluvinage, que coinciden en varios señalamientos. Los objetivos de aprendizaje y las actividades propuestas se organizaron mediante una trayectoria hipotética de aprendizaje. El análisis y la evaluación detallada de las respuestas nos permitió identificar ventajas didácticas del diseño, dificultades de aprendizaje y el importante rol que jugó la tecnología en el proceso de aprendizaje.

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INTRODUCTION

Proportional reasoning is crucial when making decisions in daily life, for example, investment decisions, comparing cost-benefit ratios, choosing among products, mixing materials or ingredients, making proportional distributions, among an endless number of options in which it is significant to think proportionally. Proportional reasoning is an example of a mathematical tool needed to interact in everyday life (Freudenthal, 1991) and, in this sense, Lamon (2020) estimates that 90% of adults do not have developed proportional reasoning. Like most mathematical concepts, proportional reasoning contains mathematical terms no less complex, such as fraction, ratio and proportion (Weiland *et al*., 2021); which are studied from elementary to higher education, and at all levels problems have been detected in their learning. In addition, it should be noted that its application is transversal to other sciences, such as physics, chemistry and economics (Lamon, 2007).

Due to the relevance of the subject, in a brief state of the art we have grouped the problem of proportional reasoning in four directions:

1) Excessive *arithmetization*: the analysis of the relationship between variables is replaced by the application of an algorithm - usually rule of three or cross product - without reasoning the covariation processes (Lobato *et al*., 2010).

2) Poor development of proportional skills: the application of mathematical rules without meaning for students leads to the creation of habits and limits their application to problems of a specific type, since there is no opportunity to solve problems in different contexts that allow exploring the multiple representations of proportional situations (Weiland *et al*., 2021).

3) Disregard of the concept of ratio in the curriculum: in most current curricula, fractions are prioritized as representatives of rational numbers, which leads to a rigid and narrow view of mathematical space, since ratios involve magnitudes of two measurement spaces; the learning of ratios is relegated and students are expected to discover for themselves the difference between ratios and fractions (Confrey & Carrejo, 2005).

4) Difficulty in distinguishing linear relationships from non-linear relationships: a tendency has been detected to apply the model of direct proportionality in contexts where it is not applicable. This problem, often called "illusion of linearity" (De Bock *et al*., 2007), has generated debate as to whether this phenomenon is generated by an epistemological obstacle or whether it is caused by inadequate teaching of the concept of proportionality (Modestou & Gagatsis, 2010).

Many of the aforementioned studies have been conducted in environments without digital technology, where pictorial figures are often used and associated problems are solved using pencil and paper. According to the educational standards of the National Council of Teachers of Mathematics (NCTM) (2000), technology is essential in the teaching and learning of mathematics, since it directly influences the ways of teaching and enhances learning.

In Mexico, studies such as the one by Balderas *et al*. (2014) have reported the dependence of both teachers and students on the rule of three algorithm, in whose application a deep-rooted omission to analyze the variables involved has been observed. The Mexican school curriculum (SEP, 2017) intends that from elementary level to third year of high school (14-15 years) a proportional thinking is consolidated that makes it possible to face any problem of a proportional nature, which we consider possible in a teaching ecosystem that follows up the progress of students and the connection between the topics involved.

The path begins with the teaching of fractions and multiplicative exercises in the third and fourth years of primary school (8-9 years); in the fifth and sixth years, the pre-proportional stage begins (10-11 years), in which ratios are compared and their equivalence is studied, and simple tables of multiplicative ratios are generated. The proportional stage begins in the first year of secondary school (12-13 years old), where direct proportionality is learned and work is done with variation tables and percentages. By the second year of secondary school (13-14 years), direct proportionality should be consolidated, inverse proportionality and proportional distributions are introduced. Finally, in the third year of secondary school (14-15 years) covariation, similarity of polygons and linear functions are studied, with which students are expected to have developed advanced proportional reasoning.

We believe that there is no conscious linkage of the topics of proportional reasoning in mathematics classrooms; however, developing such a judgment exceeds the limitations of this work, so we adhere to the search for solutions that contribute to form students with the expected learning.

In considering the above issues, the question facing the community is how to promote sound proportional reasoning in students. Our position regarding how knowledge is produced in the classroom starts from a socio-constructivist premise: the view of learning as an active construction implies that students organize and modify their current forms of mathematical knowledge, which are influenced by their sociocultural environment. Therefore, an approach to the diverse ways in which learners interpret particular mathematical situations is crucial for instructional design and teaching (Cobb *et al*., 1992).

According to the above, our proposal aims to confront the problems described through didactic activities with a functional approach in realistic contexts and supported by digital technology. The hypothesis is that students' proportional reasoning will become increasingly sophisticated if skills that confront the four problems posed are promoted.

By using digital technology, students can explore more representations and examples than is possible in traditional teaching, therefore, ways of reasoning emerge that are complex to observe without the use of technology. However, the NCTM (2000) warns that digital technology is not a panacea and that its use can be both adequate and deficient. In turn, technology can be used in two ways: as a cognitive tool and as an instrument of numerical and algebraic calculation. In this study we have used it in the first way, that is, as a tool that helps to build concepts and develop skills.

Duijzer *et al*. (2017) conducted an investigation with elementary students through an application on a multitouch tablet, in which it was found that students managed to coordinate sight and gestures associated with a proportional relationship. For her part, Gueudet (2007) conducted a teaching experiment with high school students using a web platform (mathenpoche.sesamath.net) with problems of proportional situations related to similarity, comparison of ratios and equivalence. The author reports that although students' proportional reasoning skills increased, they faced the drawback that the platform only shows the answers, in addition to the fact that students had little written participation. These drawbacks may be generated because the platforms are often not designed for specific didactic purposes, are not flexible in their adaptation to educational contexts and do not take into account the synergy that should exist between the design of tasks and technological resources.

Likewise, Lobato and Thanheiser (2002) used SimCalc MathWorlds, which helped students in their understanding of proportional situations. The authors recommend that instruction with the use of technology be combined with pencil and paper work to obtain better results in student learning. On the same line, different projects for the use of digital technologies in public elementary schools have been implemented in Mexico; for example, Red Escolar, Enseñanza de las Matemáticas con Tecnología (EMAT), Enciclomedia, Habilidades Digitales para Todos, Mi Compu mx and @prende.mx, projects developed in some states of the country for different school grades (Padilla-Partida, 2018); it is worth mentioning that none of these were consolidated.

THEORETICAL FRAMEWORK

Realistic mathematics

The realist mathematics education (REM) teaching approach emphasizes that "realistic situations" are fundamental for learning this subject. According to this current, mathematics begins in reality, understanding it as a historical and cultural construction (Freudenthal, 1991). Consequently, if mathematics education starts from situations that are meaningful to students, then they have the opportunity to attribute meaning to the mental constructions they develop while solving problem situations. In this way, their knowledge gradually becomes more formal and less dependent on the initial context. Therefore, students are active participants in the development of their own learning and in the construction of models that mathematize reality from an everyday context (Gravemeijer, 2020).

Two types of mathematization are identified in MRE: horizontal mathematization, where students move from the real to the symbolic in order to respond to problems in their own context, and vertical mathematization, where students make conceptual connections and create strategies to solve problems within the mathematical system, i.e., they separate from the context towards the path of abstraction and generalization (Van den Heuvel-Panhuizen & Drijvers, 2020).

According to Freudenthal (1991), the context and didactic design should allow students to move from horizontal mathematization to vertical mathematization. Gravemeijer (2020) identifies four levels towards vertical mathematization: 1) situational level: reality is interpreted and organized through informal and context-dependent mathematical reasoning (horizontal mathematization); 2) referential level: schemas and models that make sense within the initial context are created, vertical mathematization begins when "models of. ."; 3) general level: concepts are related, strategies that are separated from the context are generated, reasoning takes place in the mathematical world and "models for..." emerge; and 4) formal level: concepts are understood through their mathematical symbolism, reflection has moved to the mathematical world and models can be dispensed with.

Proportional reasoning

Regarding our object of study, we start from the definition of Lesh *et al.* (1988): "Proportional reasoning is the ability to work with situations involving variation, change, a sense of covariation and multiple comparison, as well as the ability to mentally process and store several pieces of information" (p. 93). In line with this definition, Modestou and Gagatsis (2010) developed a model of proportional reasoning that can be described in three categories: analogical reasoning, proportionality, and meta-analogical awareness (see Figure 1). For the present research we affiliate ourselves with this proposal and agree that inadequate teaching contributes to the epistemological obstacle of linearity.

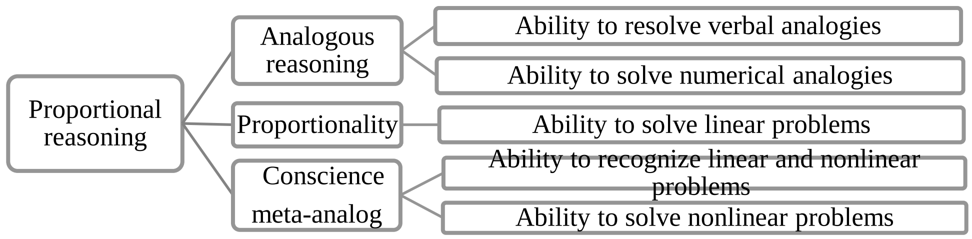

Figure 1. Proportional reasoning model

Fuente: adaptado de Modestou y Gagatsis (2010, p. 39).

Regarding what proportional reasoning involves, we identify proportional skills as those faculties that are indicated as necessary for a person to possess sound proportional reasoning. Lobato *et al.* (2010) and Weiland *et al.* (2021) indicate that these skills should include: 1) attending to and coordinating two quantities that vary dependently, 2) recognizing and using the structures of proportional situations (equivalence of ratios, constant of proportionality and linearity), 3) understanding proportionality from multiple representations (symbolic, algebraic, tabular and graphical), and 4) distinguishing linear from nonlinear situations. We consider these four skills to be key to our study and name them proportional skills.

Muttaqin *et al.* (2017) mention that to develop proportional reasoning skills in students, teachers can propose ratio and proportion tasks in a broad context that allows students to experiment, discuss, and make predictions; furthermore, tasks should help them connect proportional reasoning to other processes they already execute or understand. In our task design we take into account these teaching recommendations to promote the four proportional reasoning skills outlined above.

Design-based research

To encompass the theoretical framework and guide the intervention we relied on the design-based research (DBR) approach proposed by Bakker (2018). The cyclical nature of the approach is based on three phases: 1) preparation and design, 2) teaching experiment (implementation), and 3) retrospective analysis and redesign. Because in this type of research design and innovation in the classroom are key aspects, the use of digital technology is proposed to allow students to interact with multiple representations, facilitate the simulation of realistic situations, and test their own results in interactive virtual teaching environments (IVDEs).

To organize the design of tasks that guide students in the mathematization process, we rely on the hypothetical learning trajectory (HLT) (Simon, 2020). In turn, in the design of the activities that integrate each task we rely on the Cuevas-Pluvinage didactic framework, which provides a didactic engineering that rescues didactic principles of Piaget and the active school, adapting them to mathematics education (Cuevas & Pluvinage, 2003).

Regarding the use of technology, we found that one of the most complex tasks for a teacher is to implement constructivist didactic principles in a traditional classroom with the usual elements. For example, introducing a mathematical concept in a traditional classroom through the problematization of an everyday situation is almost impossible. A possible solution is to have a virtual scenario that simulates a real phenomenon, where students interact with various contexts dynamically and its use allows the construction of mathematical knowledge (Moyer-Packenham & Bolyard, 2016).

In this way, students are given the opportunity for action and to learn at their own pace. Digital technology endows the student and the teacher with a kind of portable laboratory by using various devices, such as cell phones and tablets (Cuevas *et al.*, 2017). The exposed problematic and the previous theoretical considerations lead us to the following research question: what advantages (or disadvantages) can be seen in the proportional reasoning of students when they are presented with proportionality tasks, in realistic contexts mediated by technology, focused on the distinction between linear and nonlinear situations?

METHODOLOGY

Based on our methodological framework, the IBD approach used consists of the following phases: preparation and design, teaching experiment, and retrospective analysis.

Preparation and design phase

The following were designed and developed: 1) a pre-test to obtain a diagnosis of the mathematical prerequisites to address the topics to be studied; 2) a THA that guides the teaching process and sets the learning objectives; 3) three sequences of didactic activities with their respective EDVIs "Orangeade", "Zoom Totoro" and "Cars"; and 4) a survey to detect possible instrumentation problems. The pre-test and final survey were hosted on Google Forms to be answered at home in order to save classroom time.

Sequence of activities (tasks)

Three EDVIs were designed in GeoGebra, with their respective exploration and guided learning (HEAG) sheets. For the design, it was considered that the virtual environments should be of adaptable proportions for viewing on different devices. The sequence of activities can be seen in the didactic path shown in Figure 2.

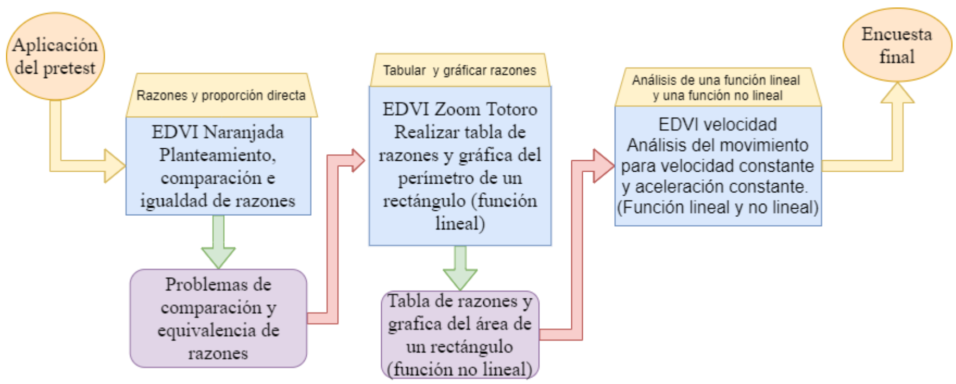

Figure 2. Design of the didactic path that guides the sequence of instruction.

Fuente: elaboración propia.

Task 1: the objective is to compare proposals for mixing orange juice and water (see figure 3a and 3b). The objective is for students to learn to pose, compare and determine equivalence between ratios, as well as to generate tables of equivalent ratios and identify the constant of proportionality. Finally, an extra activity is performed to apply the knowledge learned in different contexts.

Task 2: a zoom effect must be performed to reduce or increase an image of figures of the character Totoro according to a similarity ratio that is entered in an input box (see figure 3c). The objective is for students to develop similarity activities using the constant of proportionality. It is also intended to address the problem of the illusion of linearity by tabulating and graphing the ratio-perimeter (linear) and ratio-area (quadratic) relationships, in order to lead students towards a comparison of the two models.

Task 3: consists of visualizing a car moving at a constant speed above the allowed limit and a patrol car that starts a chase with constant acceleration as the car passes by (see figure 3d). The objective is for students to interact with the EDVI as they describe, analyze, and compare the characteristics and representations of the two motions, i.e., uniform rectilinear motion (URM) and uniformly accelerated rectilinear motion (UARM). In this way, students must identify the models present (linear and quadratic) in order to compare them and move through their tabular, graphical, algebraic and symbolic representations.

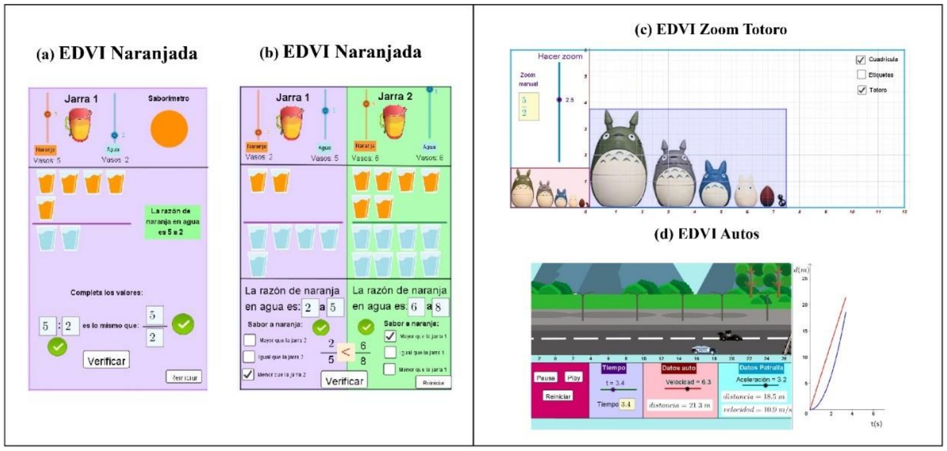

Figure 3. EDVI used in the teaching sequence.

Teaching experiment phase

The study was conducted through a face-to-face intervention in two groups of a technical high school in Mexico. Thirty-five students, aged fourteen and fifteen, participated in the study in the winter of 2021. Instruction was divided into three 90-minute sessions in a school computer room. Each student was provided with a personal computer that had the EDVIs preloaded, and they also had their respective HEAGs in printed form. The intervention was conducted by the author of this article, supported by the group's teacher and a research assistant. Collaborative learning was promoted in the sessions, and responses were discussed as a group.

RESULTS

In the analysis of the results, we evaluated the progress of the students in the levels of mathematization based on the proportional reasoning skills cited in our theoretical framework. Table 1 shows the articulation of the skills evaluated with the levels proposed by the MRE. The following should be considered in the criteria used: 1) the formal level was discarded because its scope does not correspond to the educational level of the students and the activities were not designed to reach this level; 2) we use the term proportionality to refer to direct proportionality unless otherwise indicated; 3) the graphical representation describes a functional relationship in a Cartesian plane; and 4) the criteria associated with each level do not reflect the students' competence to solve specific problems, but only categorize the characteristics of the skills that are developed from the contexts presented.

Table 1. Criteria developed to evaluate tasks based on the articulation between proportional reasoning skills and EMR mathematization levels

Proportional Reasoning Skills	Level 1: Situational	Level 2: Referential	Level 3: General
Attend to and coordinate two amounts that vary depending on the	They perceive the covariation of one variable with respect to another in a specific context	They identify the variation and dependence between two variables in a context in order to make predictions or inferences about the change of one of them	They determine the type of functional relationship between two variables by identifying the dependent variable and the independent variable
Recognize and use the structures of proportional situations	Pose and compare reasons from a context using intuitive reasoning	They use equivalence as a means of comparing ratios and identify and operate with the proportionality constant	They perceive proportionality as a linear model y = kx and understand the cross-multiplication algorithm to compare ratios
Understanding Proportionality from Multiple Representations	They capture discrete data from a linear covariation context to generate a tabular representation or graphical representation; They can make inferences from representations within the same context	They can determine the linear property of a phenomenon from tabular or graphical representations and obtain the algebraic model of the particular situation	They associate linearity with a discrete multiplicative relationship in the tabular representation, a continuous straight line in a graphical representation, and an equation of the form y = kx+a in the symbolic representation
Distinguish linear from non-linear situations	Distinguish linearity (or lack thereof) by taking or analyzing discrete data from a situation of covariation in tabular or graphical representations	They relate linearity to a constant multiplicative factor in the tables and a proportional covariation in the graphs	They compute and explicitly identify the slope in linear functions and the ratio of nonlinear change in nonlinear functions regardless of the given representation

The results obtained from the 17 students who completed all the tasks are presented below.

Task 1: "Orangeade".

In this task students showed skills in posing ratios in different notations, comparing ratios, applying the multiplicative principle to obtain equivalent ratios, and using the unit ratio method to solve practical problems. By performing the orangeade activities, students used their prior knowledge in a horizontal mathematical process, correctly assimilating three equivalent notations for a ratio (a is b, a:b, a/b). They all noted that due to a random orangeade proposal, the intensity of flavor depends on a covariation process, thus reflecting the situational level, i.e., their mathematical activity occurs within the numerical world only to interpret the context and make a judgment, horizontal mathematization is a going back and forth from the context to the mathematical world.

A finding that contributes to answering the research question was generated during the activities of this task; the EDVI presented some students with a pair of orangeade proposals in which the difference between the antecedent and the consequent was the same in both ratios, for example 2:7 and 3:8 (see Figure 4a). An intuitive way that students used to compare the ratios was manifested at that time, which consisted of mentally performing a 1-to-1 bijection between glasses of orange and glasses of water to make a decision about the flavor based on the leftover glasses. In the example in Figure 4a, once the bijection was done, in both scenarios the same number of glasses of water remained unassociated, so students responded that the two have the same taste, i.e., the same proportion.

They extended this misconception to all situations and it worked for them when the remaining glasses of water were different, except for the equivalent ratios. However, when the remaining glasses of water were equal, the EDVI pointed out the error, incomprehensible to some of the students. At the close of the session, through a group discussion, methods for comparing ratios were specified and it was found that most of the students adopted the reasoning described, and some were even surprised that the method failed. Because the EDVI delivers random propositions, this misconception was detected and the students had the need to change the way they interpreted and compared ratios.

The activity of completing tables of equivalent ratios is subject to the context (orange-water mixtures in the same proportion): the persistence of additive rather than multiplicative reasoning was observed in some students (see Figure 4b). One way to compare equivalent ratios from tables is to consider pairs of ratios that have equal numerators or denominators;

students made use of that fact (see Figure 4c) to make numerical judgments and give a covariation argument based on the denominator of the ratios.

In the application exercises following the EDVI activity, 43% of the students efficiently used the unit ratio method to compare ratios. Figure 4d shows a ratio ordering exercise (adapted from Lamon, 2020), which consists of arranging a set of animals in relation to their weight and the amount of food they consume. The unit ratio method for solving problems in different contexts is located at the referential level of the ability to recognize and use proportional structures.

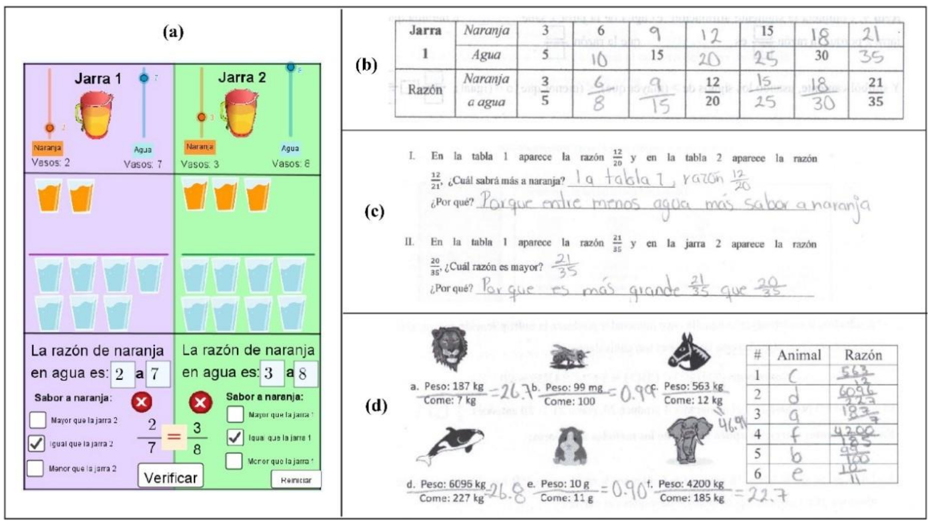

Figure 4. Evidence of activities correspondi

Task 2: "Zoom Totoro".

Students showed the ability to transition between tabular, graphical and algebraic representations of proportionality, obtaining a model that describes the perimeter of similar rectangles. The initial stage of the task involves processes of visualization, measurement and the multiplicative principle. The activity of calculating the dimensions of the EDVI image for different ratios is anchored to the context (situational level); however, following the pattern leads the way to the referential level, since students noticed that to obtain the measures of the replicated image, the measures of the original image must be multiplied by the similarity ratio (see table in Figure 5a).

To calculate the perimeter of enlarged images according to a similarity ratio, students relied on the table of dimensions, moving away from the context because the initial concept is related, in the first instance, to the perimeter and, in the second, to the area. Figure 5a corresponds to a student who calculates the perimeters for the ratios proposed in the table, manages to arrive at the linear expression P = 9x and applies the particular model found; moreover, he transits correctly between the tabular and graphical representations, recognizing the linear characteristics of the context.

It is important to point out that, although 75% of the participants correctly completed the table and the perimeter graph (see figures 5a and 5b), it was subsequently identified that only 18% recognized the linear characteristic of the variation of the perimeter with respect to the similarity ratio, so that the linear characteristics of the representations were not generally associated with the concept of perimeter. As for the transit between the representations of the ratio-area relationship, only 25% of the students completed the table and made the graph (see Figure 5c), but none obtained a correct algebraic model. Consequently, students did not have the opportunity to contrast between the linear property of perimeter and quadratic property of area. In general, the students did not go beyond the situational level in the activities of this task due to the arithmetic difficulties and their deficiencies regarding the concepts of perimeter and area.

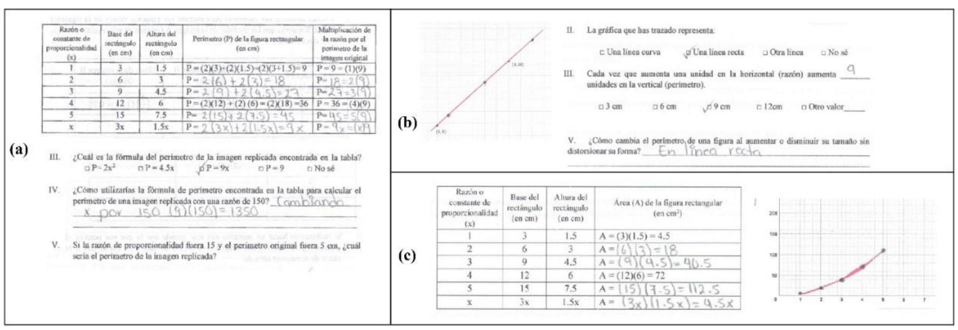

Figure 5. Evidence of the activities corresponding to task 2.

Task 3: "Cars".

Students evidenced their ability to distinguish a linear situation from a non-linear one. In general, they identified the distinctive features of each in the different representations. Initially, the activities in this EDVI are of instrumentation and contextualization. All students encountered the characteristics of motion at both constant velocity and constant acceleration. Their informal ideas about these concepts were modified by interacting with the EDVI, by pointing out the characteristics of the context variables, and by making the observations requested in the task activities.

From the context, all students completed the indicated covariation tables; of these, only 60% correctly described linear motion and only 25% showed signs of finding a variation pattern for acceleration (see Figures 6a and 6b). This is because moving from the context to the tabular representation, and eventually from the tabular representation to the graphical one, is intuitive for the students, and most of them even recognize the linear characteristics of the situation. It should be noted that recognizing the pattern of variation to explicitly express a general algebraic model is a task that requires more abstraction.

In this sense, 81% of the students correctly graphed both movements and compared them (see Figure 6c); however, only 65% reached conclusions of a general nature (see Figure 6d), so it is clear that making the leap to the general level requires a greater accompaniment of the activities. At this point in the intervention, clear progress was observed in the students' proportional reasoning skills. In this last task, 65% of the students identified relationships between variables that go beyond the initial context, concluding that the variation is proportional (in the car) and quadratic (in the patrol car).

Figure 6d shows how one student interprets movement and variation by arguing that, if equal distances are traveled in equal times, distance and time have a constant relationship, and adds that the growth of acceleration will always exceed that of speed because "the speed of the car does not change and that of the patrol car is constantly increasing." In these reasonings we observe a conceptual enrichment that transcends the context; the manipulations are not numerical or algebraic, they are of a general nature, that is, they are reasonings that apply to any context of MRU and MRUA. In this respect, Freudenthal (1991) mentions that "the relationship between constant ratio and linearity is a feat of vertical mathematization" (p. 43).

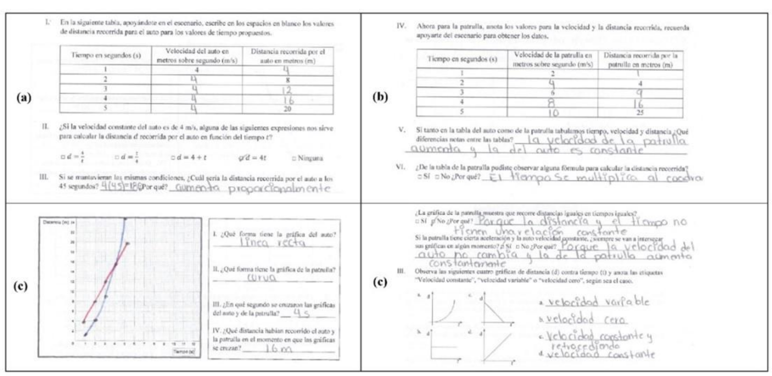

Figure 6. Evidence of the activities corresponding to task 3.

Table 2 shows the summary of the analysis of results based on the criteria established in the methodology. In Table 2 the X characters mean that, due to the design of the task, it is not possible to evaluate that skill at the corresponding level. We observed an increase in the percentage of success of the students in the skills evaluated from task 1 to task 3, so we assume that proportional reasoning was promoted in the skills considered. Nevertheless, the analysis revealed aspects to be improved in the design of tasks and corrections in the EDVIs that allow students greater freedom for meaningful learning.

Table 2. Summary of the results obtained in each of the tasks, with percentage of students who reached each level of mathematics

Proportional Reasoning Skills	Task 1: Orangeade (%)			Task 2: Zoom Totoro (%)			Task 3: Cars (%)		
	N1	N2	N3	N1	N2	N3	N1	N2	N3
Attend to and coordinate two amounts that vary depending on the	100	43	X	100	87	0	100	100	60
Recognize and use the structures of proportional situations	100	43	X	100	93	0	100	87	62
Understanding Proportionality from Multiple Representations	93	X	X	75	18	0	81	68	25
Distinguish linear from non-linear situations	X	X	X	25	25	0	87	56	0

DISCUSSION

In task 1, students had difficulties in comparing two reasons whose difference between antecedent and consequent was the same; this idea changed during the session, so we infer that there are obstacles in comprehension that emerge with this type of activities. According to Sierpinska (2004), "tasks should be able to reveal students' most hidden misconceptions" (p. 14). Since this situation was present in most of the students, we consider that the use of the EDVI favored the detection of the misconception and encouraged them to question their initial intuition, which might not have happened in a static environment.

Although the strategy used by the students is erroneous, it could be interpreted as an antecedent to the informal but correct "norming" strategy, which Lamon (2020) has defined as that which is used as a unit to reinterpret a second ratio; however, in this case the students did not use it to compare the complete ratio but elements of it, which leads to the erroneous, in this case, additive principle.

In task 2, the objective was to confront the illusion of linearity, regarding the confusion presented by the students when assuming that perimeter, area and volume have a linear relationship with the similarity ratio. In light of the results, we perceived that the task seemed too directed and with little freedom to create alternative solutions, so in the redesign we considered necessary to add to the EDVI different geometric figures, even irregular ones as suggested by de Bock *et al*. (2007); for example, when analyzing the necessary amount of ink to paint a figure of Santa Claus in which its height has been increased with respect to a scale drawing.

As further practice, culturally rich contexts can be used, namely, the NCTM (2000) proposes stories such as Gulliver's Travels to make analogies between perimeter, area and volume. In this way, students will have a greater opportunity to distinguish the linear relationship of perimeter with the quadratic relationship of area and the cubic relationship of volume.

In task 3, the general reasoning arrived at by the students set the tone to deepen the activities that bring out the generalization in the algebraic representation and identify the rate of change as a measure of covariation that allows distinguishing linear situations from nonlinear ones. At this stage students showed clear progress in distinguishing and characterizing the context of velocity from a dynamic proportionality perspective, which has been defined by Miyakawa and Winsløw (2009) as a functional dependence between two variables of the form $y = mx$, where there is an input and an output; while they identify static proportionality by the expression $a:b = c:d$.

In our teaching experiment the objective is to bridge the gap between proportional reasoning (associated with static proportionality) and linear functions (associated with dynamic proportionality). In this type of task, proportional, functional and algebraic reasoning are related; therefore, the importance of proportional reasoning as a prerequisite to access algebra is corroborated (Hemmi *et al*., 2021).

CONCLUSIONS

Proportional reasoning is a mathematical concept that transcends school grades (from elementary to high school), and constitutes a background for understanding advanced mathematical concepts of differential and integral calculus, differential equations, linear algebra, probability and statistics, among others. It is identified that when trying to promote proportional reasoning in high school, obstacles appear, such as excessive mechanization in the teaching of mathematics and the lack of mathematical skills of implicit concepts such as fraction, ratio and proportion.

In this sense, we try to find ways of teaching by analyzing the concepts involved in proportional reasoning. First with approaches towards ratio and proportion and, subsequently, through linear functions and their representations (symbolic, algebraic, tabular and graphical). Therefore, the axis of the proposal is the concept of reason as a precursor to access equivalence, comparison and ratio tables, which leads to linear functions. It is intended that students experiment with realistic situations, both linear and nonlinear, and question linearity every time they face a problem. To achieve this, digital technology was a key element in simulating realistic contexts, facilitating dynamic representations and providing feedback for student learning.

According to this article, proportional reasoning is a complex problem in educational practice at all levels; hence the need to innovate didactic practices with the support of digital technology, which requires an appropriate research methodology, such as IBD. Likewise, the adoption of an explicit didactic framework was fundamental for the design of tasks and the educational software produced. Framework prescriptions such as interactivity, data randomization, feedback, inverse operations, and freedom to formulate a solution were important and allowed the young people to elaborate an experimental task with the mathematical activities.

The analysis of the results showed the advantages of working with ratios, moving through proportional representations and contrasting them with nonlinear phenomena in different contexts. We obtained sufficient evidence to discern which of the tasks, and in what way, should be deepened in order to achieve that students show better progress in the levels of mathematization. That is, the diversity of contexts favored horizontal mathematization (situational level), but considering the times in the curriculum, when redesigning the didactic sequence, we must discern what is most appropriate to the objectives, since reducing the number of contexts could increase the opportunity to achieve greater depth in the vertical mathematization process. In this sense, we deduce that fully mathematized activities are also necessary, without reference to any context that accounts for the formal level.

Although the number of participants in this study was small, the findings described in the discussion may be transferable to similar educational contexts, especially the advantages provided by technology. To be replicated, the students' prior knowledge should be taken into account, since not knowing the necessary aspects to perform the activities -for example, locating points on the plane or knowing how to operate with fractions- could bias the data analysis and not evaluate the skills proposed in the study.

With this work we find that the assessed skills of proportional reasoning are essential for higher mathematics, although a mitigating factor may be that students do not develop sufficient mastery of rational numbers to face problems of proportionality with arithmetic difficulty, but they can manage mathematically in a specific context.

This study, beyond showing how proportional reasoning can be developed, highlights the advantages of using digital technology to meet this challenge. The use of EDVI made it possible to present random data to the student, interact with the contexts dynamically, show the mathematical objects in different representations and validate the results experimentally. In addition, it made it possible to observe misconceptions that would probably go unnoticed in traditional forms of teaching.

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